Introduction to the Standard Model
William and Mary PHYS 771 Spring 2014

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Class information, including syllabus and homework assignments can be found at
`http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/`

Homework Assignment 1

1. [**5 pts.**] We are primarily using the "mostly minus" metric, $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. With this metric, the field strength tensor for a classical electromagnetic field is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{1}$$

which can be compactly expressed as $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk} B_k$.

Solutions: It makes sense to solve (b) first, then (a). To begin, we note, in the mostly plus metric,

$$A^\mu = (\phi, \vec{A})^T\,, \qquad \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^T\,,$$
$$A_\mu = (-\phi, \vec{A})^T\,, \qquad \partial^\mu = \left(-\frac{\partial}{\partial t}, \vec{\nabla}\right)^T\,.$$

From here, we can determine:

(b) Express the space-time F_{0i} and space-space F_{ij} components in terms of E_i and B_i.
[**3 pts.**]*Solution:*

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 \\ &= \frac{\partial}{\partial t} A_i - \nabla_i(-\phi) \\ &= -E_i \end{aligned}$$

And the space-space components are

$$\begin{aligned} F_{ij} &= \nabla_i A_j - \nabla_j A_i \\ &= \epsilon_{ijk} B_k \end{aligned}$$

(a) Beginning with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, derive the form of $F_{\mu\nu}$ if we work with the "mostly plus" metric, $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

[2 pts.] *Solution:* From (b), we can immediately read off

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{2}$$

2. **[10 pts.]** We discussed using the covariant derivative to construct the field strength tensors for gauge theories, $igF_{\mu\nu} = [D_\mu, D_\nu]$. Suppose we have a fermion that is a doublet that transforms as

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \psi(x) \to e^{i\alpha^a(x)t^a}\psi(x), \quad \text{with } t^a = \frac{\sigma^a}{2}, \quad \sigma_a = \text{Pauli matrices} \tag{3}$$

such that the covariant derivative is

$$D_\mu = \partial_\mu + igA_\mu(x) \qquad\qquad A_\mu(x) = t^a A^a_\mu(x) \tag{4}$$

(a) Derive the field strength tensor. You may find it useful to determine the components $igF^a_{\mu\nu} = [D_\mu, D_\nu]^a$ instead of $F_{\mu\nu} = F^a_{\mu\nu}t^a$.
[5 pts.] *Solution:*

$$\begin{aligned}[] [D_\mu, D_\mu] &= [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu] \\ &= ig(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]) \end{aligned}$$

The commutator of A-fields is

$$\begin{aligned}[] [A_\mu, A_\nu] &= A^a_\mu A^b_\nu [t^a, t^b] \\ &= A^a_\mu A^b_\nu i\epsilon^{abc} t^c \end{aligned}$$

and so the field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g\epsilon^{abc} t^c A^a_\mu A^b_\nu \,.$$

We can project onto the a-th component using $\mathrm{tr}[t^a t^b] = \delta^{ab}/2$, and after relabeling dummy indices, we can write

$$F^a_{\mu\nu} \equiv 2\mathrm{tr}[t^a F_{\mu\nu}] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \,.$$

(b) In terms of the A^a fields, what is the form of the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}[F^2_{\mu\nu}] = -\frac{1}{4}(F^a_{\mu\nu})^2 = ? \tag{5}$$

[5 pts.] *Solution:*

$$\begin{aligned} (F^a_{\mu\nu})^2 &\equiv F^a_{\mu\nu} F^{a,\mu\nu} \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu)(\partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu} - g\epsilon^{ade} A^{d,\mu} A^{e,\nu}) \end{aligned}$$

Relabeling the dummy indices, using the symmetry properties of ϵ^{abc} and the equality $\epsilon^{abc}\epsilon^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}$, we find

$$\begin{aligned} \mathcal{L} = -\frac{1}{4}\Big[&(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu}) - 4g\epsilon^{abc} A^b_\mu A^c_\nu \partial^\mu A^{a,\nu} \\ &+ g^2 (A^a_\mu A^{a,\mu} A^b_\nu A^{b,\nu} - A^a_\mu A^{a,\nu} A^{b,\mu} A^b_\nu) \Big] \end{aligned}$$

3. **[5 pts.]** For a classic electromagnetic field, Eq. (1),

   (a) What is $F_{\mu\nu}F^{\mu\nu} = ?$
   [2 pts.] *Solution:*

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= -2E_i^2 - \epsilon_{ijk}B_k(-\epsilon^{ijl}B_l) \\ &= 2(\mathbf{B}^2 - \mathbf{E}^2) \end{aligned}$$

   (b) What is $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = ?$ (with the convention $\epsilon^{0123} = +1$)
   [3 pts.] *Solution:* With this convention, we can replace $\epsilon^{0ijk} = \epsilon^{ijk}$, $\epsilon^{i0jk} = -\epsilon^{ijk}$ etc. We then have

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= \epsilon^{ijk}(F_{0i}F_{jk} - F_{i0}F_{jk} + F_{ij}F_{0k} - F_{ij}F_{k0}) \\ &= 2\epsilon^{ijk}(F_{0i}F_{jk} + F_{ij}F_{0k}) \\ &= 4\epsilon^{ijk}F_{0i}F_{jk} \\ &= 4E_i\epsilon^{ijk}(-\epsilon^{jkh}B_h) \\ &= -8\mathbf{E}\cdot\mathbf{B} \end{aligned}$$

4. **[5 pts.]** For $U(\lambda) = e^{i\lambda\alpha_a X_a}$ where X_a are the generators of a Lie Algebra,

   (a) show $U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2)$
   Solution: The crucial step is to realize $[\alpha_a X_a, \alpha_b X_b] = 0$ which is easy to show

$$\begin{aligned} [\alpha_a X_a, \alpha_b X_b] &= \alpha_a\alpha_b[X_a, X_b] \\ &= \left(\frac{1}{2}\{\alpha_a, \alpha_b\} + \frac{1}{2}[\alpha_a, \alpha_b]\right)[X_a, X_b] \\ &= \frac{1}{2}\{\alpha_a, \alpha_b\}[X_a, X_b] \\ &= 0 \end{aligned}$$

   which all follows from symmetry/anti-symmetry. The vanishing commutation relation implies

$$e^{i\lambda_1\alpha_a X_a}e^{i\lambda_2\alpha_a X_a} = e^{i(\lambda_1+\lambda_2)\alpha_a X_a}$$

   and hence $U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2)$.

5. **[5 pts.]** For $SU(2)$, what is the matrix form of the generators
   Solution: For any representation, we know the "ladder" operators span the space

$$J_\pm|jm\rangle = c^\pm_{jm}|jm \pm 1\rangle\,, \qquad c^\pm_{jm} = \sqrt{j(j+1) - m(m\pm 1)}$$

   and that

$$[J_+, J_-] = 2J_3\,, \qquad J_\pm = J_1 \pm iJ_2\,.$$

   These ladder operators can be easily constructed in a given representation by beginning with the lowest/maximum state, and raising/lowering it. The above relations can then be used to construct the generators.

(a) for the $j = 1$ representation?

[3 pts.] *Solution:*

$$J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} , \qquad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} ,$$

from which we get

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

(b) for the $j = 3/2$ representation?

[2 pts.] *Solution:*

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} ,$$

from which we get

$$J_3 = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} , \quad J_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} , \quad J_2 = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} .$$

6. **[10 pts.]** Dirac algebra. In any representation, the Dirac matrices satisfy the algebra (in 4 dimensions)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{1}_{4\times 4} . \tag{6}$$

In class, we defined the Dirac matrices in the "Dirac Basis", for which

$$\gamma_D^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} , \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \quad \gamma_D^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} , \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 . \tag{7}$$

Another useful and very common basis is the "chiral basis" (or Weyl basis) in which

$$\gamma_\chi^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} , \qquad \gamma_\chi^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \qquad \gamma_\chi^5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} , \tag{8}$$

(a) Determine the similarity transformation which converts from the Dirac to chiral basis

$$\gamma_\chi = S\gamma_D S^{-1} \qquad\qquad S = ? \tag{9}$$

[4 pts.] *Solution:* This is simply a matter of diagonalizing γ_D^5 and using the eigenvectors to construct the rotation matrix. The only trick is to make sure we preserve the sign convention of the $\gamma_D^i = \gamma_\chi^i$ matrices. One finds

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(b) What is the similarity transformation that transforms from the chiral to Dirac basis?

[2 pts.] *Solution:* This is simply given by S^{-1}, as $\gamma^5_D = S^{-1}S\gamma^5_D S^{-1}S$:

$$S^{-1} = S^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(c) In both the Dirac and chiral basis, in terms of the spinor components, what are

$$\psi_\pm = \frac{1 \pm \gamma^0}{2}\psi = ? \tag{10}$$

[2 pts.] *Solution:* If we write $\psi^T = (\psi_1, \psi_2, \psi_3, \psi_4)$ then we have

$$\text{Dirac}: \qquad \psi_+ = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \qquad \psi_- = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\text{Chiral}: \qquad \psi_+ = \frac{1}{2}\begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \\ \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix} \qquad \psi_- = \frac{1}{2}\begin{pmatrix} \psi_1 - \psi_3 \\ \psi_2 - \psi_4 \\ -\psi_1 + \psi_3 \\ -\psi_2 + \psi_4 \end{pmatrix}$$

(d) In both the Dirac and chiral basis, in terms of the spinor components, what are

$$\begin{aligned} \psi_R &= \frac{1 + \gamma^5}{2}\psi = ? \\ \psi_L &= \frac{1 - \gamma^5}{2}\psi = ? \end{aligned} \tag{11}$$

[2 pts.] *Solution:*

$$\text{Dirac}: \qquad \psi_R = \frac{1}{2}\begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \\ \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix} \qquad \psi_L = \frac{1}{2}\begin{pmatrix} \psi_1 - \psi_3 \\ \psi_2 - \psi_4 \\ -\psi_1 + \psi_3 \\ -\psi_2 + \psi_4 \end{pmatrix}$$

$$\text{Chiral}: \qquad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$$

7. **[35 pts.]** In class, we discussed the g-factor for the electron and the nucleons. We saw in general, the elastic electromagnetic structure of a fermion, with parity conserving interactions, can be expressed as

$$\bar{u}(p')\Gamma^\mu(p', p)u(p) = \bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p), \qquad q = p' - p, \tag{12}$$

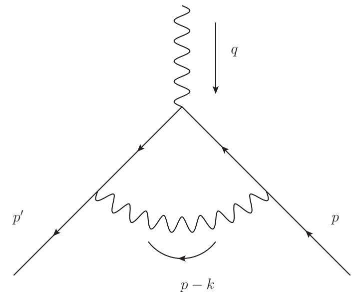


FIG. 1: The Feynman diagram used to compute $g-2$ of a point fermion.

where $u(p)$ is an on-shell fermion spinor which satisfies $\not{p}u(p) = mu(p)$ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. This is the "elastic" structure, because $\bar{u}(p')$ also represents an on-shell fermion satisfying $\bar{u}(p)\not{p} = \bar{u}(p)m$.

In the case of the electron (point-like fermion) we saw the Dirac equation gives $g = 2 + \frac{\alpha_{f.s.}}{\pi}$. We noted that for the nucleons, $g_p \simeq 5.58$ and $g_n = -3.83$ so that the nucleons are not perturbatively close to point-like fermions, one indication they have interesting internal structure. We commented in class that $g = 2[F_1(0) + F_2(0)]$ and so for the electron, $F_2(0) = \frac{\alpha_{f.s.}}{2\pi}$.

Perform this classic QED calculation, using tools you have been learning in QFT, see Fig. 1. This calculation is so classic, you can easily find the solution in the literature. *I strongly encourage you to attempt it on your own, before resorting external sources, peers, books, etc.*. The key to successfully performing this calculation is to realize you isolate the contribution which is proportional to $\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)$. It turns out, this contribution to the diagram in Fig. 1 is free of both Ultraviolet (UV) ($q_E^2 \to \infty$) and Infrared (IR) ($q_E^2 \to 0$) singularities (where q_E is the Euclidean four-momentum obtained after Wick rotation of the momentum integral). To this end, recall the Gordon Identity which can be used to relate $\bar{u}(p')(p'+p)^\mu u(p)$ to $\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)$.

(a) Compute $g-2$ for the electron

[**20 pts.**]*Solution:* See attached hand notes.

(b) Using just the requirements we have of our QFT, QED (renormalizable, gauge-invariant, Lorentz invariatn QFT in 4 space-time dimensions) why should you know ahead of time that the contribution to $g-2$ is free of both UV and IR singularities?

[**5 pts.**] *Solution: UV* See attached hand notes.

[**10 pts.**] *Solution: IR* See attached hand notes.

e^- g-2 to 1-loop in QED

The most general current between on-shell spinors is

$$\bar{u}(p')\,\Gamma_\mu(p',p)\,u(p)$$

$$\Gamma_\mu(p',p) = \gamma_\mu F_1(-q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(-q^2)$$

$$q \equiv p' - p \quad , \quad \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

The gyromagnetic coefficient is defined as

$$g \equiv 2\left[F_1(0) + F_2(0)\right]$$

Current conservation gives

$$F_1(0) = 1$$

We are interested in computing the correction, which is the coefficient of $\frac{i\sigma_{\mu\nu}q^\nu}{2m}$

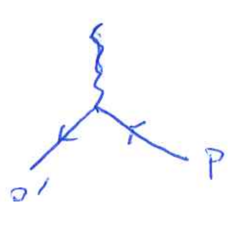
To determine this quantity, we can couple the electron to a classical electromagnetic field with

$$\delta\mathcal{L} = -\hat{e}\, J_\mu A^\mu_{cl.} \quad , \quad J_\mu = \bar{\Psi}\gamma_\mu\Psi$$

The matrix element for an electron interacting with this external field is then given at leading order by

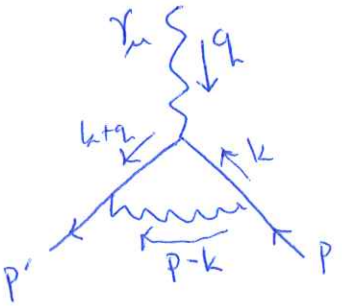
$$\langle e, p' | iA | e p\rangle = \bar{u}(p')(-i(-\hat{e}))\gamma_\mu u(p) A^\mu_{cl} \quad , \quad \hat{e}\,u(p) = -e\,u(p)$$

$$= -ie\,\bar{u}(p')\gamma_\mu u(p) A^\mu_{cl.}$$



We want to compute the "radiative correction"

The factor of $-ie$ and A^{μ}_{cl} will be the same for the one loop correction. So we can consider the correction to $\Gamma_\mu(p',p) = \gamma_\mu + O(e^2)$



$$k' \equiv k+q$$

$$\langle ep'|iA|ep\rangle = -ieA^{\mu}_{cl.}\ \bar{u}(p')\int\frac{d^4k}{(2\pi)^4}(-ie)\gamma_\rho\frac{-ig^{\rho\sigma}}{(p-k)^2+i\varepsilon}\frac{i}{\not{k}'-m+i\varepsilon}\gamma_\mu\frac{i}{\not{k}-m+i\varepsilon}(-ie)\gamma_\sigma\,u(p)$$

We can simplify the fermion propagators, $\dfrac{i}{\not{k}-m+i\varepsilon} = \dfrac{\not{k}+m}{k^2-m^2+i\varepsilon}$

and just look at the correction to Γ_μ

$$\Rightarrow \bar{u}(p')\,\delta\Gamma_\mu\,u(p) = (-ie)^2(i)^3(-)\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\gamma^\rho(\not{k}'+m)\gamma_\mu(\not{k}+m)\gamma_\rho\right]u(p)}{\left[(k-p)^2+i\varepsilon\right]\left[k'^2-m^2+i\varepsilon\right]\left[k^2-m^2+i\varepsilon\right]}$$

$$= (-)^3(i)^5e^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\gamma^\rho\not{k}'\gamma_\mu\not{k}\gamma_\rho + m(\gamma^\rho\not{k}'\gamma_\mu\gamma_\rho+\gamma^\rho\gamma_\mu\not{k}\gamma_\rho)+m^2\gamma^\rho\gamma_\mu\gamma_\rho\right]\times u(p)}{\left[(k-p)^2+i\varepsilon\right]\left[k'^2-m^2+i\varepsilon\right]\left[k^2-m^2+i\varepsilon\right]}$$

We can use the 4-d identities

$$\gamma^\rho\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho = -2\gamma_\nu\gamma_\mu\gamma_\sigma$$
$$\gamma^\rho\gamma_\mu\gamma_\nu\gamma_\rho = 4g_{\mu\nu}$$
$$\gamma^\rho\gamma_\mu\gamma_\rho = -2\gamma_\mu$$

$$\bar{u}(p')\,\delta\Gamma_\mu u(p) = -ie^2\int d^4\!\not{k}\;\frac{\bar{u}(p')\left[-2\not{k}\gamma_\mu\not{k}' + 4m(k_\mu+k'_\mu) - 2m^2\gamma_\mu\right]u(p)}{\left[(k-p)^2+i\varepsilon\right]\left[k'^2-m^2+i\varepsilon\right]\left[k^2-m^2+i\varepsilon\right]}$$

$\not{k} = \frac{k}{2\pi}$

Note: we have not included a photon mass (for IR regularization) nor have we worried about d-dimensional properties of Dirac Algebra. This is from the benefit of hindsight that the piece of this vertex we ~~have~~ are interested in is free of both IR and UV divergences.

To proceed with the integral, we can either combine the denominators with the Feynman Parameter trick

$$\frac{1}{a_1 a_2 \cdots a_n} = \int_0^1 dx_1\, dx_2 \cdots dx_n\; \delta\Big(1-\sum_{i=1}^{n} x_i\Big)\frac{(n-1)!}{\left[x_1 A_1 + x_2 A_2 + \cdots x_n A_n\right]^n}$$

Or we can use the Schwinger Integral trick

$$\frac{1}{k^2-m^2+i\varepsilon} = \frac{1}{i}\int_0^\infty dz\; e^{iz(k^2-m^2+i\varepsilon)}$$

For fun, lets first solve with the Schwinger Integral trick.

$$\bar{u}(p')\,\delta\Gamma_\mu u(p) = 2ie^2(-i)^3\int d^4\!\not{k}\;\bar{u}(p')\left[\not{k}\gamma_\mu\not{k}' + m^2\gamma_\mu - 2m(k_\mu+k'_\mu)\right]u(p)\int_0^\infty dz_1 dz_2 dz_3$$
$$\times\exp\left\{i\left[z_1(k-p)^2 + z_2(k'^2-m^2) + z_3(k^2-m^2) + i\varepsilon(z_1+z_2+z_3)\right]\right\}$$

We can shift the order of integration and look for a simple momentum integration variable. Completing the square, we find the argument of the exponential can be written

$$i\left\{z_{123}\,\ell^2 - m^2\left(z_2+z_3-z_1+\frac{z_1^2}{z_{123}}\right) - q^2\left(\frac{z_2^2}{z_{123}}-z_2\right) + \frac{2z_1z_2}{z_{123}}\,p\cdot q\right\}$$

with $\ell = k - \frac{z_1}{z_{123}}p + \frac{z_2}{z_{123}}q$, $\quad z_{123} \equiv z_1+z_2+z_3$

For on-shell electrons, we also have

$$2\,p\cdot q = (p+q)^2 - p^2 - q^2$$
$$= p'^2 - p^2 - q^2$$
$$= m^2 - m^2 - q^2$$

Which allows us to reduce the argument to

$$i\left\{ z_{123}\,\ell^2 - \frac{z_2 z_3}{z_{123}}\,q^2 - \frac{(z_2+z_3)^2}{z_{123}}\,m^2 \right\}$$

Assuming we can shift integration order, we have

$$\bar{u}(p')\,\delta\Gamma_\mu\,u(p) = \cancel{-2e^2\int d^4k}$$
$$-2e^2\int_0^\infty dz_1 dz_2 dz_3 \int d^4k\;\bar{u}(p')\left[\not{k}\gamma_\mu\not{k}' + m^2\gamma_\mu - 2m(k_\mu + k'_\mu)\right]u(p)$$
$$\times \exp\left\{ i\left[z_{123}\,\ell^2 - \frac{z_2 z_3}{z_{123}}\,q^2 - \frac{(z_2+z_3)^2}{z_{123}}\,m^2 \right]\right\}$$

So, we now shift the momentum integration variable

$$= -2e^2\int_0^\infty dz_1 dz_2 dz_3 \int\frac{d^4\ell}{(2\pi)^4}\,\exp\left\{ i\left[z_{123}\,\ell^2 - \frac{z_2 z_3}{z_{123}}\,q^2 - \frac{(z_2+z_3)^2}{z_{123}}\,m^2 \right]\right\}$$
$$\times\,\bar{u}(p')\Bigg[\left(\not{\ell} + \frac{z_1}{z_{123}}\not{p} - \frac{z_2}{z_{123}}\not{q}\right)\gamma_\mu\left(\not{\ell} + \frac{z_1}{z_{123}}\not{p} + \frac{z_1+z_3}{z_{123}}\not{q}\right) + m^2\gamma_\mu$$
$$- 2m\left(2\ell_\mu + 2\frac{z_1}{z_{123}}p_\mu + \frac{z_1-z_2+z_3}{z_{123}}q_\mu\right)\Bigg]u(p)$$

The terms odd in ℓ will integrate to ϕ, so we can drop them

$$= -2e^2\int_0^\infty dz_1 dz_2 dz_3\; e^{-i\left\{\frac{z_2 z_3}{z_{123}}q^2 + \frac{(z_2+z_3)^2}{z_{123}}m^2\right\}}\int\frac{d^4\ell}{(2\pi)^4}\,e^{i z_{123}(\ell^2 + i\varepsilon)}\;\times$$
$$\bar{u}(p')\Bigg\{\not{\ell}\gamma_\mu\not{\ell} + \left(\frac{z_1}{z_{123}}\not{p} - \frac{z_2}{z_{123}}\not{q}\right)\gamma_\mu\left(\frac{z_1}{z_{123}}\not{p} + \frac{z_1+z_3}{z_{123}}\not{q}\right) + m^2\gamma_\mu$$
$$- 2m\left[2\frac{z_1}{z_{123}}p_\mu + \frac{z_1-z_2+z_3}{z_{123}}q_\mu\right]\Bigg]u(p)$$

We need to simplify the numerator structure keeping in mind the Gordon Identity between on-shell spinors

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p')\left[\frac{p'_\mu + p_\mu}{2m} + \frac{i\sigma_{\mu\nu}q^\nu}{2m}\right]u(p)$$

$$\not{l}\gamma_\mu\not{l} = \not{l}\, l^\alpha \gamma_\mu \gamma_\alpha$$
$$= \not{l}\, l^\alpha \left(2 g_{\mu\alpha} - \gamma_\alpha \gamma_\mu\right)$$
$$= \not{l}\left(2 l_\mu - \not{l}\gamma_\mu\right)$$
$$= 2 l_\alpha l_\mu \gamma^\alpha - \not{l}\not{l}\gamma_\mu$$

under the $d^4 l$ integral, we can simplify these. $l_\alpha l_\mu \to \frac{1}{4} g_{\alpha\mu} l^2$

$$\Rightarrow \frac{1}{2} l^2 g_{\alpha\mu}\gamma^\alpha - l^2\gamma_\mu$$
$$= -\frac{1}{2} l^2 \gamma_\mu$$

$$\left(\frac{z_1}{z_{123}}\not{p}' - \frac{z_2}{z_{123}}\not{q}\right)\gamma_\mu\left(\frac{z_1}{z_{123}}\not{p} + \frac{z_1+z_3}{z_{123}}\not{q}\right)$$

$$= \left(\frac{z_1}{z_{123}}\not{p}' - \frac{z_1+z_2}{z_{123}}\not{q}\right)\gamma_\mu\left(\frac{z_1}{z_{123}}\not{p} + \frac{z_1+z_3}{z_{123}}\not{q}\right)$$

from $\bar{u}(p')\not{p}' = \bar{u}(p')m$

$\not{p}\, u(p) = m\, u(p)$

$$\Rightarrow \left(\frac{z_1}{z_{123}} m - \frac{z_1+z_2}{z_{123}}\not{q}\right)\gamma_\mu\left(\frac{z_1}{z_{123}} m + \frac{z_1+z_3}{z_{123}}\not{q}\right)$$

we have $\not{q}\gamma_\mu = (\not{p}'-\not{p})\gamma_\mu$

$$= m\gamma_\mu - \not{p}\gamma_\mu$$
$$= m\gamma_\mu - 2p_\mu + \gamma_\mu\not{p}$$
$$= 2m\gamma_\mu - 2p_\mu$$

similarly

$$\gamma_\mu\not{q} = 2p'_\mu - 2m\gamma_\mu$$

also, between spinors,

$$\not{q}\gamma_\mu\not{q} = -q^2\gamma_\mu$$

$$= \gamma_\mu\left[\frac{z_1+z_2}{z_{123}}\frac{z_1+z_3}{z_{123}} q^2 - \frac{3z_1^2 + 2z_1(z_2+z_3)}{z_{123}^2} m^2\right] + 2m\frac{z_1}{z_{123}^2}\left[z_{12}\, p_\mu + z_{13}\, p'_\mu\right]$$

This gives us

$$\bar{u}(p')\,\delta\Gamma_\mu\, u(p) = -2e^2\int_0^\infty dz_1 dz_2 dz_3\; e^{-i\left\{\frac{z_2 z_3}{z_{123}}q^2 + \frac{(z_2+z_3)^2}{z_{123}}m^2\right\}}\int\frac{d^4 l}{(2\pi)^4}\, e^{i z_{123}\{l^2+i\varepsilon\}}$$

$$\times\,\bar{u}(p')\Bigg[-\frac{1}{2}l^2\gamma_\mu + \gamma_\mu\left[m^2\,\frac{-2z_1^2+(z_2+z_3)^2}{z_{123}^2} + q^2\,\frac{(z_1+z_2)(z_1+z_3)}{z_{123}^2}\right]$$

$$+(p'_\mu+p_\mu)\left[-m\,\frac{z_1(z_2+z_3)}{z_{123}^2}\right]$$

$$+(p'_\mu-p_\mu)\left[m\,\frac{(z_2-z_3)(z_1+2(z_2+z_3))}{z_{123}^2}\right]\Bigg]\,u(p)$$

Notice, the term proportional to $q_\mu = p'_\mu - p_\mu$ integrates to zero as it is odd under the interchange of $z_2 \longleftrightarrow z_3$, while the integration measure is even. We can use the Gordon Identity to identify the contribution to F_2. As we are only interested in this correction, we can ignore the rest

$$\Rightarrow \bar{u}(p')\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}\,\delta F_2\, u(p) = \bar{u}(p')\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}\,u(p)$$

$$\times -2m\;\times(-2)e^2 m\int_0^\infty dz_1 dz_2 dz_3\; e^{-i\left\{\frac{z_2 z_3}{z_{123}}q^2+\frac{(z_2+z_3)^2}{z_{123}}m^2\right\}}\;\frac{z_1(z_2+z_3)}{z_{123}^2}$$

$$\times\int\frac{d^4 l}{(2\pi)^4}\, e^{i(z_{123}(l^2+i\varepsilon))}$$

The gaussian integral is easy (after Wick rotation)

$$\int\frac{d^4 l}{(2\pi)^4}\, e^{i z_{123}(l^2+i\varepsilon)} = i\int\frac{d^4 l_E}{(2\pi)^4}\, e^{-z_{123}(l_E^2)} = \frac{i}{(2\pi)^4}\left(\sqrt{\frac{\pi}{z_{123}}}\right)^4$$

$$= \frac{i}{16\pi^2}\,\frac{1}{z_{123}^2}$$

$$\bar{u}(p')\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}\,\delta F_2\, u(p) = +\frac{2ie^2m^{\times 2m}}{16\pi^2}\int_0^\infty dz_1dz_2dz_3\;\frac{z_1(z_2+z_3)}{(z_1+z_2+z_3)^4}\;e^{-i\left\{\frac{z_2z_3}{z_{123}}q^2+\frac{(z_2+z_3)^2}{z_{123}}m^2\right\}}$$

$$\times\;\bar{u}(p')\,i\sigma_{\mu\nu}q^\nu/2m\;u(p)$$

We are further only interested in the value at $q^2=0$.
Notice the oscillatory nature of the exponential. We could make the change $m^2\to m^2-i\varepsilon$, recalling how the $i\varepsilon$ appears in the propagator. This leaves us with

$$\bar{u}(p')\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}\,\delta F_2(q^2\to 0)\,u(p) = \frac{+2i\,\alpha_{f.s.}\,m^{\times 2m}}{4\pi}\int_0^\infty dz_1dz_2dz_3\;\frac{z_1(z_2+z_3)}{(z_1+z_2+z_3)^4}\;\exp\left\{\left(-im^2-\varepsilon\right)\frac{(z_2+z_3)^2}{z_1+z_2+z_3}\right\}$$

$$\times\;\bar{u}(p')\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}\,u(p)$$

$$\Rightarrow\;\delta F_2(0) = +\frac{i\,m^2\alpha}{\pi}\int_0^\infty dz_1dz_2dz_3\;\frac{z_1(z_2+z_3)}{(z_1+z_2+z_3)^4}\;e^{-i(m^2-i\varepsilon)\frac{(z_2+z_3)^2}{z_{123}}}\;,\qquad z_{123}=z_1+z_2+z_3$$

insert the identity $\displaystyle 1=\int_0^\infty\frac{d\lambda}{\lambda}\,\delta\!\left(1-\frac{z_{123}}{\lambda}\right)$

$$=+i\,m^2\frac{\alpha}{\pi}\int_0^\infty dz_1dz_2dz_3\;\frac{z_1(z_{23})}{z_{123}^4}\int_0^\infty\frac{d\lambda}{\lambda}\,\delta\!\left(1-\frac{z_{123}}{\lambda}\right)e^{-i(m^2-i\varepsilon)\frac{z_{23}^2}{z_{123}}}$$

rescale all $z_i\to\lambda z_i$

$$=+i\,m^2\frac{\alpha}{\pi}\int_0^\infty dz_1dz_2dz_3\;\frac{z_1(z_{23})}{z_{123}^4}\int_0^\infty\frac{d\lambda}{\lambda}\cdot\lambda^3\cdot\frac{\lambda^2}{\lambda^4}\,\delta(1-z_{123})\;e^{-i\lambda(m^2-i\varepsilon)\frac{z_{23}^2}{z_{123}}}$$

$$=+i\,m^2\frac{\alpha}{\pi}\int_0^\infty dz_1dz_2dz_3\int_0^\infty d\lambda\;\delta(1-z_{123})\;\frac{z_1z_{23}}{z_{123}^4}\;e^{-i\lambda(m^2-i\varepsilon)\frac{z_{23}^2}{z_{123}}}$$

$$=+i\,m^2\frac{\alpha}{\pi}\int_0^\infty dz_1dz_2dz_3\left\{\delta(1-z_{123})\,\frac{z_1z_{23}}{z_{123}^4}\;\frac{z_{123}}{z_{23}^2(-i)(m^2)}\;\underbrace{e^{-i\lambda(m^2-i\varepsilon)\frac{z_{23}^2}{z_{123}}}\Big|_0^\infty}_{-1}\right\}$$

$$\delta F_2 = +\frac{\alpha}{\pi}\int_0^\infty dz_3 \int_0^\infty dz_2 \int_0^\infty dz_1 \; \delta(1 - z_{123}) \; \frac{z_1}{z_{23}\, z_{123}^4}$$

$$= +\frac{\alpha}{\pi}\int_0^1 dz_3 \int_0^{1-z_3} dz_2 \; \frac{1 - z_2 - z_3}{z_2 + z_3}$$

$$= +\frac{\alpha}{\pi}\int_0^1 dz_3 \left[-1 + z_3 - \ln z_3\right]$$

$$\delta F_2 = +\frac{\alpha}{2\pi}$$

$$g = 2\left[F_1(0) + F_2(0)\right]$$

$$g - 2 = 2 F_2(0)$$

$$= \frac{\alpha}{\pi}$$

Let us also use the Feynman Parameter trick – it's always good to have multiple ways to compute things

$$\bar{u}(p)\,\delta\Gamma_\mu u(p) = 2ie^2 \int d^4k\, \bar{u}(p')\left[\not{k}\gamma_\mu \not{k}' + m^2\gamma_\mu - 2m(k_\mu + k'_\mu)\right]u(p)$$
$$\times \int_0^1 dx \int_0^{1-x} dy \frac{2}{\left[x(k-p)^2 + y(k'^2-m^2) + (1-x-y)(k^2-m^2) + i\varepsilon\right]^3}$$

where we have already made use of the $\delta(1-x-y-z)$ to perform the dz integral. We want to simplify the denominator by completing the square

$$x(k-p)^2 + y(k'^2-m^2) + (1-x-y)(k^2-m^2) + i\varepsilon$$

$$= k^2(x+y+1-x-y) - 2x\,k\cdot p + xp^2 + y2k\cdot q + yq^2 - m^2(y+1-y-x) + i\varepsilon$$

$$= k^2 - 2xk\cdot p + 2yk\cdot q + yq^2 - m^2(1-x) + xm^2 + i\varepsilon \quad , \quad p^2 = m^2$$ on-shell external e^-

$$= \underbrace{(k - xp + yq)^2}_{\ell} - x^2p^2 - y^2q^2 + 2xy\,p\cdot q + yq^2 - m^2(1-2x) + i\varepsilon$$

$$= \ell^2 - y(y-1)q^2 + 2xy\,p\cdot q - m^2(1-x)^2 + i\varepsilon$$

$$2p\cdot q = (p+q)^2 - p^2 - q^2$$
$$= p'^2 - p^2 - q^2$$
$$= -q^2$$

$$= \ell^2 - y(y-1)q^2 - xyq^2 - m^2(1-x)^2 + i\varepsilon$$

$$= \ell^2 - q^2y(x+y-1) - m^2(1-x)^2 + i\varepsilon$$

$$= \ell^2 - y(1-x-y)(-q^2) - m^2(1-x)^2 + i\varepsilon \quad , \quad q^2 \leq 0$$ for the problem we are solving

We need to shift the numerator also

$$k = l + xp - yq \qquad \int\frac{d^4 l}{(2\pi)^4}$$

$$\bar{u}(p')\,\delta\Gamma_\mu u(p) = 4ie^2\int_0^1 dx\int_0^{1-x} dy\,\frac{\bar{u}(p')\left[\,(\not{l}+x\not{p}-y\not{q})\gamma_\mu(\not{l}+x\not{p}+(1-y)\not{q}) + m^2\gamma_\mu - 2m(2l_\mu + 2xp_\mu - 2yq_\mu + q_\mu)\right]u}{\left[l^2-\Delta\right]^3}$$

all odd powers of l integrate to 0

$$\Delta = y(1-x-y)(-q^2) + m^2(1-x)^2 - i\varepsilon$$

$$= 4ie^2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4 l}{(2\pi)^4}\,\frac{\bar{u}(p')\left[\not{l}\gamma_\mu\not{l} + (x\not{p}-y\not{q})\gamma_\mu(x\not{p}+(1-y)\not{q}) + m^2\gamma_\mu - 2m\left(2xp_\mu + q_\mu(1-2y)\right)\right]u(p)}{\left[\,l^2-\Delta\,\right]^3}$$

Now we need to manipulate the numerator structure to isolate the term of interest.

$$\not{l}\gamma_\mu\not{l} = \not{l}\,l^\alpha\gamma_\mu\gamma_\alpha$$
$$= \not{l}\,l^\alpha(2g_{\mu\alpha} - \gamma_\alpha\gamma_\mu)$$
$$= 2\not{l}\,l_\mu - \not{l}\not{l}\gamma_\mu \quad :$$
$$= -\frac{1}{2}l^2\gamma_\mu$$

$$\left\{\begin{aligned} \not{l}\not{l} &= l^\alpha l^\beta\left[\tfrac{1}{2}\{g_{\alpha\beta}\gamma_\alpha,\gamma_\beta\} + \tfrac{1}{2}[\gamma_\alpha,\gamma_\beta]\right] \\ &= l^\alpha l^\beta\, g_{\alpha\beta} \\ &= l^2 \\ 2\not{l}\,l_\mu &= 2l_\alpha l_\mu\,\gamma^\alpha \\ &= 2\cdot\tfrac{1}{4}g_{\alpha\mu}\,l^2\gamma^\alpha \\ &= \tfrac{1}{2}l^2\gamma_\mu \end{aligned}\right.$$

We want terms proportional to $\sigma_{\mu\nu}q^\nu$ but we have to be careful, as the Gordon Identity relates

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p')\left[\frac{p'_\mu + p_\mu}{2m} + \frac{i\sigma_{\mu\nu}q^\nu}{2m}\right]u(p)$$

so we also have to find the terms proportional to p, p'

We can also use $\quad \not{p}\,u(p) = m\,u(p)$

$$\bar{u}(p')\not{p}' = m\bar{u}(p')$$

$$\bar{u}(p')\,\delta\Gamma_\mu u(p) = 4ie^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i\varepsilon]^3}$$

$$\times\,\bar{u}(p')\left[-\frac{1}{2}\ell^2\gamma_\mu + m^2\gamma_\mu + \underbrace{(x\not{p}-y\not{q})\gamma_\mu(x\not{p}+(1-y)\not{q})} - 2m(2x\,p_\mu + q_\mu(1-2y))\right]u(p)$$

$$(x\not{p}'-(x+y)\not{q})\gamma_\mu(x\not{p}+(1-y)\not{q})$$

$$=(xm-(x+y)\not{q})\gamma_\mu(xm+(1-y)\not{q})$$

$$\cancel{=(xm-(x+y)2m)\gamma_\mu(xm+(1-y)(-2m))}$$

$$= m^2x^2\gamma_\mu - xm(x+y)(2m\gamma_\mu - 2p_\mu)$$

$$+xm(1-y)(2p'_\mu - 2m\gamma_\mu)$$

$$-(x+y)(1-y)(-q^2)\gamma_\mu$$

$$=\gamma_\mu\left[m^2x^2 - 2m^2x(x+y) - 2m^2x(1-y) + q^2(x+y)(1-y)\right]$$

$$+2p_\mu\left[xm(x+y)\right]$$

$$+2p'_\mu\,xm(1-y)$$

$$=\gamma_\mu\left[m^2x^2 - 2m^2x(x+1) + q^2(x+y)(1-y)\right]$$

$$+2p_\mu\,xm(x+y) + 2p'_\mu\,xm(1-y)$$

$$= 4ie^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i\varepsilon]^3}$$

$$\times\,\bar{u}(p')\Big[\gamma_\mu\left[-\frac{1}{2}\ell^2 + m^2\left(1+x^2-2x(x+1)\right) + q^2(x+y)(1-y)\right]$$

$$+p_\mu\left[2m\,x(x+y) - 4mx + 2m(1-2y)\right] \qquad 2m\left[x(x+y) - 2x + 1 - 2y\right]$$

$$+p'_\mu\left[\underbrace{2mx(1-y) - 2m(1-2y)}\right]\Big]u(p)$$

$$2m\left[x(1-y) - 1 + 2y\right]$$

$$Ap + Bp' = A'(p'+p) + B'(p'-p)$$

$$= p'(A'+B') + p(A'-B')$$

$$A = A' - B' \qquad A' = \frac{1}{2}(A+B)$$

$$B = A' + B' \qquad B' = \frac{1}{2}(B-A)$$

$$\bar{u}(p')\,\delta\Gamma_\mu\,u(p) = 4ie^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^3}$$

$$\times\,\bar{u}(p')\Big[\gamma_\mu\left[-\tfrac{1}{2}\ell^2+m^2(1-x)^2+q^2(x+y)(1-y)\right]$$

$$+2m(p'_\mu+p_\mu)\cdot\tfrac{1}{2}x(x-1)$$

$$+2m(p'_\mu-p_\mu)\,\tfrac{1}{2}(2-x)(2y-1+x)\Big]u(p)$$

We can use the Gordon Identity to replace

$$p'_\mu+p_\mu \rightarrow 2m\gamma_\mu - 2m\,\frac{i\sigma_{\mu\nu}q^\nu}{2m}$$

As we are interested in the correction proportional to $\frac{i\sigma_{\mu\nu}q^\nu}{2m}$,

we can just focus on this terms.

$$\Rightarrow \delta F_2 = 4ie^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^3}\,(-2m^2)\,x(x-1)$$

$$= +8im^2e^2\int_0^1 dx\,x(1-x)\int_0^{1-x}dy\,\underbrace{\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^3}}_{\frac{-i}{(4\pi)^2}\frac{1}{2}\frac{1}{\Delta}}\,,\qquad \Delta = m^2(1-x)^2-q^2y(1-x-y)$$

$$= m^2\frac{\alpha}{\pi}\int_0^1 dx\,x(1-x)\underbrace{\int_0^{1-x}dy\,\frac{1}{m^2(1-x)^2-q^2y(1-x-y)}}$$

This integral is finite and free of IR ($q^2=0$) divergences. It is related to a log (arcCotn). We will just focus on the $q^2=0$ part

$$= \frac{\alpha}{\pi}\int_0^1 dx\,x(1-x)\cdot\frac{1}{1-x}$$

$$\delta F_2 = \frac{\alpha}{2\pi}$$

$$\Rightarrow\ g-2 = 2\delta F_2 = \frac{\alpha}{\pi}$$

Why is g-2 free of UV divergences @ 1-loop?

- Simply stated, the operator that would be needed to absorb a UV divergence would be

$$\mathcal{L} = e\bar{\Psi}\,\frac{i\sigma_{\mu\nu}F^{\mu\nu}}{2m}\,\psi$$

to have the correct Lorentz structure to be able to cancel a divergence. However, assuming QED is a renormalizable QFT, we know this operator is not allowed as it is dimension - 5. Without such an operator, there can be no UV divergence, or the theory is not correct or not "renormalizable".

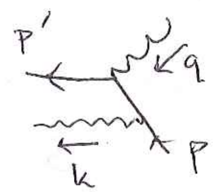
Why is g-2 free of IR divergence @ 1-loop?

Generically, IR divergences are associated with long-range physics. Because we can understand the classical world with classical physics, with our QFT description of the world, there can not be such an IR divergence, else our classical EM description would not work.

Specifically for g-2 @ 1-loop in QED, we can show with perturbation theory, there can not be an IR divergence for g-2. IR divergences arise from "soft photon radiation."

At LO in QED, the IR divergence comes from two graphs (in addition to vertex correction)

Lets look at the Lorentz structure of the first term



The IR divergence will come when k becomes co-linear with p' or p ($p\cdot k = 0$ or $p'\cdot k = 0$)

k_mu = Q (1,0,0,1)
so we can also let Q —> 0
to get the divergence

$$iA = \bar{u}(p')\,\gamma_\mu \frac{i}{\not p - \not k - m + i\varepsilon}\,\gamma_\nu\, \varepsilon^{*\nu}(k)\, u(p)$$

$$= i\,\frac{\bar{u}(p')\,\gamma_\mu(\not p - \not k + m)\,\not\varepsilon^*\, u(p)}{(p-k)^2 - m^2 + i\varepsilon}$$

$$\not p \not\varepsilon^* = -\not\varepsilon \not p + 2\varepsilon\cdot p$$

$$\cancel{\not\varepsilon^* \not k = 0}$$

$$= i\,\frac{\bar{u}(p')\,\gamma_\mu\left[-\not\varepsilon^*\not p + 2\varepsilon^*\cdot p - \not k\not\varepsilon^* + m\not\varepsilon^*\right]u(p)}{-2p\cdot k}$$

$$= i\,\frac{\bar{u}(p')\left[\gamma_\mu\, 2\varepsilon^*\cdot p + \gamma_\mu \not\varepsilon^* \not k\right]u(p)}{-2p\cdot k}$$

As "$k \to 0$", only the first term is IR divergent. However, this term is only proportional to $\bar{u}(p)\gamma_\mu u(p)$ and so does not have the correct Lorentz structure to contribute to $g-2$.
The same holds for the other graph